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Bulk Viscous Bianchi-III Models with Time Dependent G and Λ

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Abstract Bulk Viscous anisotropic Bianchi-III cosmological models are investigated with time dependent gravitational and cosmological constants in the framework of Einstein's general relativity. In order to get some useful information about the time varying nature of G and Λ, we have assumed an exponentially decaying rest energy density of the universe. The extracted Newtonian gravitational constant G varies with time but its time varying nature depends on bulk viscosity and the anisotropic nature of the model. The cosmological constant Λ is found to decrease with time to a small but positive value for the models.

Keywords Variable G and Λ · Bianchi-III model · Anisotropic universe · Bulk viscosity

1 Introduction

Right after Dirac's proposal for the variation of the so called fundamental constants, from his Large Number Hypothesis [1, 2], there have been many attempts to modify Einstein's general relativity. Brans and Dicke [3], in their attempt to incorporate the time varying nature of the gravitational constant, followed Mach's principle and suggested that the gravitational field is mediated by a long range scalar field ϕ that has an inverse dimension of G. For some

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reviews on G varying cosmologies, one may refer to the articles [4, 5]. Empirically, local constraints on the rate of variation of G can be derived from Lunar Laser Ranging. From the ranging data to the Viking landers on Mars, the simple time variation in the effective Newtonian gravitational constant, $\frac{\dot{G}}{G}$ could be limited to $(0.2 \pm 0.4) \times 10^{-11}$ yr^{-1} [6]. Even though the quoted errors in this estimation are much large, which may have stemmed from the lack of knowledge of the masses of asteroids, it is certain that G varies substantially with time. In recent times, G varying cosmologies are widely investigated with the hope that they may shed some light on the varying nature of gravitation [7–13].

The analysis of the Hubble diagram of distant type Ia Supernovae provides strong evidence that the universe is currently undergoing an accelerated phase of expansion [14–16]. The accelerated expansion state of universe strongly suggests that the content of the universe is largely dominated by non-baryonic matter usually referred to as dark matter. In classical Friedman models with zero curvature, the accelerated phase of expansion of the universe is generally explained by a cosmological constant. A non-zero positive cosmological constant $\Lambda \sim 2h^2 \times 10^{-56}$ cm^{-2} is required to account for the observations of the distant type Ia Supernova [14–16]. The concept of time varying cosmological constant term in Einstein's field equations stems from the old and yet unsettled problem in physics, popularly known as the cosmological constant problem [17–20], which centers around the large difference in the theoretical and observational values of the cosmological constant. The positive, small but non-zero cosmological constant requires a fine tuning problem in physics. One possible way of obtaining a small but positive cosmological constant at the present epoch is to consider an evolving cosmological constant. The evolving cosmological constant can be introduced either by just taking it at a phenomenological level or by postulating a scalar field potential [19]. Due to the coupling of dynamic degrees of freedom with matter fields of the universe the cosmological constant is believed to relax to its present small value through the expansion of universe. An evolving cosmological constant term has been considered by many authors [21–24]. Berman [25, 26], Bertolami [21, 22], Endo and Fukui [27] and Canuto et al. [28] have suggested an inverse square time varying form of the cosmological constant i.e. $\Lambda \sim t^2$ where as Arbab in his work [9] used $\Lambda = 3\beta H^2$, where β is a constant and $H$is the Hubble parameter. In some recent works Borges and Canerio [29] and Tiwari [12] have considered a cosmological term proportional to the Hubble parameter. As can be shown later in this work, the time variation of G is intimately related to the time variation of the cosmological constant Λ. If one considers a time varying cosmological constant term, then it is imperative that one should consider a time varying gravitational constant term in the usual Einstein field equations.

In recent times, anisotropic cosmological models have generated a lot of research interest. The high degree of isotropy of the 2.7 K microwave background radiation supports the Friedman-Robertson-Walker (FRW) models. However, certain measurements have indicated dipole anisotropy of the Cosmic Microwave Background radiation (CMB) [30]. While the dipole anisotropy can readily be explained by the Sun's motion relative to the background radiation, the quadrupole anisotropy is an intrinsic property of the background radiation. The observed quadrupole amplitude has a lower value than that expected from the best fit Λ-dominated cold dark matter (ΛCDM) model to the entire power spectrum since the first data of the differential microwave radiometer (CPBER/DMR) appeared in 1992 [31–33]. This anomaly was confirmed with the high resolution data provided by the Wilkinson Microwave Anisotropy Probe (WMAP) [34–37]. This low value of quadrupole anisotropy seems to be inescapable [38]. In view of these observations, it is wise to consider spatially homogeneous and anisotropic Bianchi type cosmological models. In this context, Bianchi type-III models are widely studied to investigate different aspects of the universe and these models have a significant role in the description of the universe at the early stages of evolution.

Bulk viscous models have prime roles in getting inflationary phases of the universe [39–45]. Bulk viscosity driven inflation is primarily due to the negative bulk viscous pressure giving rise to a total negative effective pressure which may overcome the pressure due to the usual gravity of matter distribution in the universe and provide an impetus to drive it apart.

In the present work, we have investigated different bulk viscous anisotropic cosmological models in Einstein's relativity with variable gravitational and cosmological constants. We have shown that the time varying nature of the cosmological constant is intimately related to the time varying nature of the gravitational constant. Instead of assuming any particular form of the time variation of G and Λ, we have tried to extract their time varying nature from different bulk viscous cosmological models. Following our earlier works [44, 45], we have assumed that the contribution of the bulk viscosity to the total pressure is proportional to the rest energy density. Such an assumption of bulk viscous pressure (may be termed as barotropic bulk viscous pressure) results in a total effective pressure which encompasses real values both in the positive and negative domain and is essential to explain the accelerated nature of expansion of the universe.

The organization of the paper is as follows: In Sect. 2, the basic equations are derived for the Bianchi-III metric. In Sect. 3, different bulk viscous cosmological models are discussed along with the physical and kinematical properties by considering an exponentially decaying rest energy density of the universe. The summary and conclusion of the present work are presented at the end in Sect. 4.

2 The Metric and the Field Equations

The anisotropic universe is considered through the Bianchi-III metric given by,

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)e^{-2hx}dy^2 + C^2(t)dz^2 \tag{1}$$

where $A(t)$, $B(t)$ and $C(t)$ are the metric potentials considered as functions of cosmic time only and h is a chosen exponent to describe the model.

With an evolving cosmological term, Einstein's field equations are expressed as,

$$R_{ij} - \frac{1}{2}g_{ij}R = -8\pi G(t)T_{ij} + \Lambda(t)g_{ij} \tag{2}$$

where R_{ij} is the Ricci Tensor, R is the Ricci scalar. Here, the Newtonian gravitational constant G is considered as a function of time. T_{ij} is the energy momentum tensor of the matter field and it satisfies the conservation law

$$T^{ij}_{;j} = 0 \tag{3}$$

where the semicolon (;) represents covariant differentiation.

In the above field equation (2), the unit is chosen in such a manner that, the value of the speed of light in vacuum c is unity, i.e. $c = 1$.

The universe is usually described by a perfect fluid distribution. In order to take into account the dissipative phenomena occurring in the cosmic fluid, we consider that the contribution to the energy-momentum tensor should also come from bulk viscosity in addition to the usual fluid. The presence of bulk viscosity in the cosmic fluid has already been recognized in connection with the observed accelerated expansion of the universe, usually known as the inflationary scenario [39–45].

In commoving coordinates, the energy–momentum tensor is given by,

$$T_{ij} = (\rho + \bar{p})u_i u_j + \bar{p} g_{ij} \tag{4}$$

where u^i are the four velocity vectors defined as $u^i = \delta^i_4$ and they satisfy the relation $u^i u_i = -1$. ρ is the rest energy density and $\bar{p}$ is the total effective pressure which includes the proper pressure p and the contribution of bulk viscosity to pressure i.e.

$$\bar{p} = p - \xi(t) u^l_{;l} \tag{5}$$

$\xi(t)$ being the time dependent bulk viscous coefficient. The total effective pressure is generally approximated as

$$\bar{p} = p - \xi(t)\theta \tag{6}$$

In (6), $\theta = u^l_{;l}$ is the scalar expansion and for the Bianchi type-III metric in (1), it is given by

$$\theta = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \tag{7}$$

The overhead dots on the metric potentials represent the ordinary time derivatives. Defining the directional Hubble parameters in X-, Y- and Z-axes as $H_1 = \frac{\dot{A}}{A}$, $H_2 = \frac{\dot{B}}{B}$ and $H_3 = \frac{\dot{C}}{C}$ respectively, the mean Hubble parameter can be written as $H = \frac{1}{3}(H_1 + H_2 + H_3)$. In terms of the mean Hubble parameter, the expansion scalar will become $\theta = 3H$ and hence the total effective pressure assumes the form

$$\bar{p} = p - 3\xi H \tag{8}$$

The field equation (2), for the Bianchi type-III metric in (1), with (4) assumes the explicit forms:

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = -8\pi G \bar{p} + \Lambda \tag{9}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} = -8\pi G \bar{p} + \Lambda \tag{10}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{h^2}{A^2} = -8\pi G \bar{p} + \Lambda \tag{11}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} - \frac{h^2}{A^2} = 8\pi G \rho + \Lambda \tag{12}$$

$$h\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right) = 0 \tag{13}$$

Equation (13) immediately suggests that for $h \neq 0$, $\frac{\dot{B}}{B} = \frac{\dot{A}}{A}$ or $H_1 = H_2$ which implies that

$$B = kA \tag{14}$$

where $k \neq 0$ is an integration constant.

Along with the conservation law $T^{ij}_{;j} = 0$, the vanishing covariant divergence of the Einstein tensor $R_{ij} - \frac{1}{2}g_{ij}R$ yields

$$-8\pi G_{;j}T_{ij} + \Lambda_{;j}g_{ij} = 0 \tag{15}$$

which can further be reduced to

$$8\pi\dot{G}\rho + \dot{\Lambda} = 0 \tag{16}$$

Equation (16) implies that the time varying nature of the Newtonian constant G is very much linked to the time dependence of cosmological constant Λ. In case, when the cosmological constant Λ is assumed to be time independent (i.e. a pure constant), then $\dot{G}\rho = 0$, which emphasizes that G is also a pure constant in a matter filled universe characterized by a non-zero energy density. In other words, whenever a time varying cosmological constant term is considered in classical cosmological models, it is necessary to take into account the simultaneous time varying nature of the Newtonian gravitational constant. Presently, the universe is undergoing an accelerated phase of expansion which can be explained through a small but positive value of cosmological constant and in order to bridge the gap between the quantum cosmological models and classical cosmological models, usually an evolving Λ is considered. This necessitates the use of simultaneous time varying G and Λ in the field equations.

The expansion scalar θ and the shear scalar σ for the metric (1) are defined as

$$\theta = u^{l}_{;l} = 2H_2 + H_3 \tag{17}$$

$$\sigma^2 = \frac{1}{2}\left[\sum_i H_i^2 - \frac{1}{3}\theta^2\right] = \frac{1}{3}(H_2 - H_3)^2 \tag{18}$$

In order that the shear scalar σ be proportional to the scalar expansion θ, we can take a linear relationship between the Hubble parameters H_2 and H_3 i.e.

$$H_2 = \mu H_3 \tag{19}$$

where μ is an arbitrary constant, usually assumed to take positive values only and it takes care of the anisotropic nature of the model. Equation (19) leads to the relationship between the metric potentials B and C as

$$B = k_1 C^{\mu} \tag{20}$$

Under this assumption, $\sigma \propto \theta$ and with the relation $H_2 = \mu H_3$, the expansion scalar θ and the shear scalar σ take the forms

$$\theta = (2\mu + 1)H_3 \tag{21}$$

$$\sigma^2 = \frac{1}{3}(\mu - 1)^2 H_3^2 \tag{22}$$

and in terms of θ,

$$\sigma^2 = \frac{1}{3}\left(\frac{(\mu - 1)^2}{4\mu^2 + 4\mu + 1}\right)\theta^2 \tag{23}$$

3 Bulk Viscous Cosmological Models

The field equations are highly non-linear in nature and therefore we require certain plausible conditions to obtain determinate solutions. For a barotropic fluid, the proper pressure p is considered to be directly proportional to the energy density i.e. $p = \omega_0 \rho$, $(0 \leq \omega_0 \leq 1)$. The contribution of bulk viscosity to the pressure may also be considered in the same footing and may be taken to be proportional to the energy density i.e. $\xi\theta = \zeta\rho$ where the proportionality constant is restricted as $\zeta \geq 0$ [44, 45]. As has been emphasized in earlier works [44, 45], such a barotropic bulk viscous pressure is required to explain the presently undergoing accelerated expansion of the universe. The combined effect of proper pressure and the contribution of bulk viscosity to pressure can be expressed as

$$\bar{p} = p - 3\xi H = (\omega_0 - \zeta)\rho = \omega\rho \tag{24}$$

where $\omega = \omega_0 - \zeta$. With this form of total effective pressure, it is possible that ω can assume any real value, may be positive or negative and hence the total effective pressure may also either be positive or negative. The negative pressure can be possible only if the contribution of the bulk viscosity to the pressure is greater than the proper pressure of the usual matter field. As such a total negative pressure provides an anti gravity effect which results in an accelerated expansion phase of the universe. The limits on the equation of state parameter ω from SN Ia data are $-1.67 < \omega < -0.62$ [46] and that from a combination of SN Ia data with CMB anisotropy and galaxy clustering statistics are $-1.3 < \omega < -0.79$ [47]. From these limits it is clear that in order to explain the accelerated phase of expansion of the universe, a negative pressure is required.

The Einstein field equations (10)–(13) can now be expressed in terms of three time varying parameters C, Λ and ρ as

$$(\mu + 1)\frac{\ddot{C}}{C} + \mu^2\left(\frac{\dot{C}}{C}\right)^2 = -8\pi G\omega\rho + \Lambda \tag{25}$$

$$2\mu\left(\frac{\ddot{C}}{C}\right) + (3\mu^2 - 2\mu)\left(\frac{\dot{C}}{C}\right)^2 - \frac{k^2h^2}{k_1^2C^{2\mu}} = -8\pi G\omega\rho + \Lambda \tag{26}$$

$$(\mu^2 + 2\mu)\left(\frac{\dot{C}}{C}\right)^2 - \frac{h^2k^2}{k_1^2C^{2\mu}} = 8\pi G\rho + \Lambda \tag{27}$$

The conservation law for Energy-momentum tensor in (3) along with (14), (20) and (24) gives

$$(2\mu + 1)(\omega + 1)\rho\left(\frac{\dot{C}}{C}\right) + \dot{\rho} = 0 \tag{28}$$

which after integration yields

$$C = \frac{k_2}{\rho^{1/\{(\omega+1)(2\mu+1)\}}} \tag{29}$$

where $k_2 \neq 0$, is an integration constant. Since C and ρ are interdependent through (29), it is clear that, the Einstein field equations (25)–(27) can be expressed to have only two independent unknown variables ρ and Λ. From these relations, it is amply clear that, the time varying nature of Λ depends on the nature of rest energy density of the universe. Since the time variation of G and Λ are intimately related, if we know the form of ρ, we can better understand the time variation of G and Λ. The universe is expanding with the growth of cosmic time and hence the rest energy density ρ should decrease with the advancement of time. In order to get some useful information about the time varying nature of the Newtonian gravitational and cosmological constants, in the present work, we assume an exponentially decreasing rest energy density ρ,

$$\rho = \rho_0 e^{-\chi t^n} \tag{30}$$

where ρ_0 is the initial value of the energy density and χ and n are positive constant parameters describing the rate of decrease of ρ with time. For such an exponentially decaying rest energy density of the universe the total effective pressure has the form $\bar{p} = \omega\rho_0 e^{-\chi t^n}$, which encompasses all possible real values of the equation of state both in the positive and negative domain.

With such an assumption of exponentially decaying rest energy density, the metric potential C becomes

$$C = \rho_2 e^{\chi_1 t^n} \tag{31}$$

where $\rho_2 = \frac{k_2}{\rho_0^{1/(\omega+1)(2\mu+1)}}$ and $\chi_1 = \frac{\chi}{(\omega+1)(2\mu+1)}$ are constants.

Consequently the other two metric potentials A and B can be expressed as

$$A = \frac{k_1}{k}\rho_2^{\mu} e^{\chi_1 \mu t^n} \tag{32}$$

$$B = k_1 \rho_2^{\mu} e^{\chi_1 \mu t^n} \tag{33}$$

The kinematical and physical parameters of the model are

$$\text{The spatial volume:}\quad V = \sqrt{-g} = ABCe^{hx} = \frac{k_1^2}{k}\rho_2^{2\mu+1} e^{(2\mu+1)\chi_1 t^n + hx} \tag{34}$$

$$\text{The Hubble parameter:}\quad H = \left(\frac{(2\mu+1)\chi_1 n}{3}\right) t^{(n-1)} \tag{35}$$

$$\text{The scalar expansion:}\quad \theta = (2\mu+1)\chi_1 n t^{(n-1)} \tag{36}$$

$$\text{The shear scalar:}\quad \sigma^2 = \frac{(\mu-1)^2 n \chi_1^2}{3} t^{2(n-1)} \tag{37}$$

$$\text{The coefficient of bulk viscosity:}\quad \xi = \frac{\zeta\rho_0}{(2\mu+1)\chi_1 n} t^{(1-n)} e^{-\chi t^n} \tag{38}$$

The interesting feature in this model is that, there is no initial singularity in the model. The universe starts with a finite volume and expands exponentially to very large values at large values of cosmic time. The time dependence of the Hubble parameter H, the scalar expansion θ and the shear scalar σ depends on the time varying nature of the rest energy density or in other words, all of them depend on the choice of the parameter n. At the beginning, when $t \to 0$, for $n > 1$, H, θ and σ all vanish and all of them increase monotonously with the increase in time. For $n < 1$, these quantities decrease with the increase in time leading to null values at infinitely large cosmic time. The evolution of the coefficient of bulk viscosity $\xi(t)$also depends on the value of n and χ. For $n > 1$, the coefficient of bulk viscosity decreases with the growth of time. For $n < 1$, as $t \to 0$, $\xi(t)$ vanishes and it increases with the increase in time. However, with the growth of time, for $n < 1$, it saturates to a constant value for small values of time and then decreases exponentially. For $n \to 1$, i.e. n becoming more and more closer to 1, the exponential factor in the expression of $\xi(t)$ dominates and helps it to vanish. In this model, we can not get a physical situation with $\omega = -1$.

3.1 Variation of G and Λ with Time

Let the left hand side of (27) be taken to be $F(C)$ so that

$$F(C) = 8\pi G\rho + \Lambda \tag{39}$$

Differentiation of (39) with respect to time and then algebraic simplification leads to

$$G = \frac{1}{8\pi\dot{\rho}}\dot{F} \tag{40}$$

where $\dot{F}$ is the time derivative of $F(C)$.

The variation of Newtonian gravitational constant G with time can be straight forwardly found from (30), (31) and (40) as

$$G = -\left[\frac{(\mu^2+2\mu)\chi_1^2 n(2n-2)t^{(n-2)}e^{\chi t^n}}{8\pi\rho_0\chi} + \frac{h^2k^2\mu\chi_1 e^{(\chi-2\mu\chi_1)t^n}}{4\pi\rho_0\chi k_1^2\rho_2^{2\mu}}\right] \tag{41}$$

and the relative growth of the Newtonian gravitational constant, $\frac{\dot{G}}{G}$ can be easily inferred from (41).

The time evolution of the cosmological constant Λ can be expressed as

$$\begin{aligned}\Lambda = {}& \left[\frac{(\mu^2+2\mu)\chi_1^2 n(2n-2)}{\chi}\right]\left(t^{(n-2)} + \frac{n\chi}{2n-2}t^{(2n-2)}\right)\\ &+ \left[\frac{2h^2k^2\mu\chi_1}{\chi k_1^2\rho_2^{2\mu}}\right](\chi-2\chi_1\mu)n\int t^{(n-1)}e^{-2\chi_1\mu t^n}dt\end{aligned} \tag{42}$$

The time evolution of the Newtonian gravitational constant and the cosmological constant depends on different factors such as μ, n, χ and ω. The cosmological constant is expressed here in integral form, included in the second term of (42). It may be mentioned here that, we can not get a clear picture of the exact time varying nature of these constants G and Λ from the' from the expressions (41) and (42) since these equations are more involved with all the factors μ, n, χ and ω. In order to get clear idea about the time evolution of these constants, it is better to take certain representative values of the parameter n namely $n = \frac{1}{2}$, $n = 1$ and $n = 2$. The choice of these values of n is completely arbitrary and is simply based on the fact that the rest energy density decays with time in a suitable manner.

(A) *Model-I*: $n = \frac{1}{2}$

For this particular case, the rest energy has the form $\rho = \rho_0 e^{-\chi\sqrt{t}}$ and the corresponding total effective pressure becomes $P = \omega\rho_0 e^{-\chi\sqrt{t}}$. In this case, the Hubble parameter, the expansion scalar and the Shear scalar become

$$H = \left(\frac{(2\mu+1)\chi_1}{6}\right)\frac{1}{\sqrt{t}} \tag{43}$$

$$\theta = \frac{(2\mu+1)\chi_1}{2}\frac{1}{\sqrt{t}} \tag{44}$$

$$\sigma^2 = \frac{(\mu-1)^2\chi_1^2}{6}\frac{1}{t} \tag{45}$$

It is clear from the above expressions (43)–(45) that the Hubble parameter, the expansion scalar and the shear scalar evolve gradually from a large value at the beginning and decrease with the increase in the cosmic time. The coefficient of bulk viscosity can be expressed as

$$\xi = \frac{2\zeta\rho_0}{(2\mu+1)\chi_1}\sqrt{t}e^{-\chi t^{1/2}} \tag{46}$$

whose time varying nature depends on the parameters χ and t.

The time variation of G and Λ can be expressed as

$$G = -\left[-\frac{(\mu^2+2\mu)\chi_1^2 t^{-3/2} e^{\chi t^{1/2}}}{16\pi\rho_0\chi} + \frac{h^2k^2\mu\chi_1 e^{(\chi-2\mu\chi_1)t^{1/2}}}{4\pi\rho_0\chi k_1^2\rho_2^{2\mu}}\right] \tag{47}$$

and

$$\Lambda = \left[-\frac{(\mu^2+2\mu)\chi_1^2}{2\chi}\right]\left(\frac{1}{t^{3/2}} - \frac{\chi}{2t}\right) - \left[\frac{h^2k^2}{\chi k_1^2\rho_2^{2\mu}}\right](\chi - 2\chi_1\mu)e^{-2\chi_1\mu t^{1/2}} \tag{48}$$

The time variation of the Newtonian gravitational constant is very much involved with the parameters μ, n, χ and ω. However, with the increase in time it decreases and vanishes for large values of cosmic time provided $\chi - 2\chi_1\mu < 0$. At the beginning of cosmic time the cosmological constant assumes a large value and it gradually decreases to a very small value with the growth of time.

(B) *Model-II*: $n = 1$

In this case the rest energy density assumes the form $\rho = \rho_0 e^{-\chi t}$. The total effective pressure for this form of rest energy density becomes $\bar{p} = \omega\rho_0 e^{-\chi t}$.

The kinematical parameters of the model are

$$\text{The spatial volume:} \quad V = \sqrt{-g} = ABCe^{hx} = \frac{k_1^2}{k}\rho_2^{2\mu+1}e^{(2\mu+1)\chi_1 t + hx} \tag{49}$$

$$\text{The Hubble parameter:} \quad H = \frac{1}{3}(2\mu+1)\chi_1 \tag{50}$$

$$\text{The scalar expansion:} \quad \theta = (2\mu+1)\chi_1 \tag{51}$$

$$\text{The shear scalar:} \quad \sigma^2 = \frac{(\mu-1)^2\chi_1^2}{3} \tag{52}$$

$$\text{The coefficient of bulk viscosity:} \quad \xi = \frac{\zeta\rho_0}{(2\mu+1)\chi_1}e^{-\chi t} \tag{53}$$

It is interesting to note that in this model, the Hubble parameter H, the scalar expansion θ and shear scalar σ are constant quantities but the volume scale factor of the model increases exponentially with time.

The expression of G, for this assumption of $n=1$, can be calculated in a straight forward manner from (41) as

$$G=-\left(\frac{h^2k^2\mu}{4\pi\rho_0\rho_2^{2\mu}k_1^2(\omega+1)(2\mu+1)}\right)e^{(\chi-2\chi_1\mu)t} \tag{54}$$

The relative growth of G now becomes $\frac{\dot{G}}{G}=\chi-2\chi_1\mu$ which comes out to be a constant quantity. The time evolution of the cosmological constant Λ can be expressed as

$$\Lambda=-\left(\frac{h^2k^2}{\rho_2^{2\mu}k_1^2\chi}\right)(\chi-2\chi_1\mu)e^{-2\chi_1\mu t}+\nu_1 \tag{55}$$

where ν_1 is an integration constant. For the sake of simplicity we may take $\nu_1=0$. As in the previous model with inverse power law behavior of the rest energy density, in the present model, the time variation of the Newtonian gravitational constant depends on the equation of state parameter ω containing the effect of bulk viscosity and the anisotropic parameter μ through the quantity $\chi-2\chi_1\mu=\chi(1-\frac{2\mu}{(\omega+1)(2\mu+1)})$. If ω is less than -1, then G comes out to be an increasing function of time. In fact, in that case, it grows exponentially with time. If ω is greater than -1, then the time evolution of G depends on the quantity $g(\omega,\mu)=\frac{2\mu}{(\omega+1)(2\mu+1)}$. If $g(\omega,\mu)<1$, then G is an increasing function of time and if $g(\omega,\mu)>1$, then G is a decreasing function of time. In case $g(\omega,\mu)=1$, then G comes out to be a pure fundamental constant. However, once again we should mention that in this model, if we take $\omega=-1$, then the model shows unphysical and unusual results. Even though some of the empirical/ extracted results suggest that $\frac{\dot{G}}{G}$ should vary inversely with time [8], here, in this model, we do not get such a result, rather we get constant value for it. This kind of deviation of our result may be attributed to the inflationary kind of solution we have assumed. However, the experimental results extracted may or may not be in accordance with the inflationary nature and therefore, the present model may have some importance in this direction. The present day observed value of cosmological constant is very small but positive i.e. $\Lambda\sim 2h^2\times 10^{-56}$ cm^{-2} is required to account for the observations of the distant type Ia Supernova [14–16], where $h=\frac{H_0}{100}$ Km s^{-1} Mpc^{-1} and H_0 is the present estimate of the Hubble parameter. Empirically, the present value of h probably lies between 0.4 and 1. From the expression of the cosmological constant in (55), a positive value of Λ can be ascertained only if $\chi-2\chi_1\mu<0$ implying $g(\omega,\mu)>1$. The possibility that $g(\omega,\mu)=1$ is also not admissible as it leads to a null value of Λ, which is not at all required for the explanation of the observations from measurements of distant type Ia supernova. With this inference of $g(\omega,\mu)>1$, Λ decays exponentially to a very small but positive value. Since $g(\omega,\mu)>1$, we can safely conclude that, in an inflationary kind of solution as presumed by taking an exponentially decaying rest energy density, the Newtonian gravitational constant also decays exponentially with time.

The critical densities ρ_c, ρ_Λ and the density parameters Ω_m, Ω_Λ for this model can be expressed as

$$\rho_c=-\frac{(2\mu+1)^2\chi_1\rho_0\rho_2^{2\mu}\chi k_1^2}{6h^2k^2\mu}e^{-(\chi-2\chi_1\mu)t} \tag{56a}$$

$$\rho_\Lambda = \left(\frac{\chi}{2\mu\chi_1} - 1\right)\rho_0 e^{-\chi t} \tag{56b}$$

$$\Omega_m = -\left[\frac{6h^2k^2\mu}{(2\mu+1)^2\chi_1\rho_2^{2\mu}\chi k_1^2}\right]e^{(-2\chi_1\mu)t} \tag{56c}$$

$$\Omega_\Lambda = -\left[\frac{3h^2k^2(\chi - 2\chi_1\mu)}{(2\mu+1)^2\chi_1^2\rho_2^{2\mu}\chi k_1^2}\right]e^{-2\chi_1\mu t} \tag{56d}$$

(C) *Model-III*: $n = 2$

With this assumption, the rest energy density assumes the form $\rho = \rho_0 e^{-\chi t^2}$ and the total effective pressure becomes $\bar{p} = \omega\rho_0 e^{-\chi t^2}$.

The kinematical parameters of the model are

The spatial volume: $V = \sqrt{-g} = ABCe^{hx} = \frac{k_1^2}{k}\rho_2^{2\mu+1}e^{(2\mu+1)\chi_1 t^2 + hx}$ (57)

The Hubble parameter: $H = \frac{2}{3}(2\mu+1)\chi_1 t$ (58)

The scalar expansion: $\theta = 2(2\mu+1)\chi_1 t$ (59)

The shear scalar: $\sigma^2 = \frac{2(\mu-1)^2\chi_1^2}{3}t^2$ (60)

The coefficient of bulk viscosity: $\xi = \frac{\zeta\rho_0}{2(2\mu+1)\chi_1}\frac{e^{-\chi t^2}}{t}$ (61)

In the beginning of cosmic time i.e. $t \to 0$, the universe starts with null values of the Hubble parameter H, the scalar expansion θ and shear scalar. But with the growth of time all of these parameters increase to become very large values as $t \to \infty$. In this case, the coefficient of bulk viscosity ξ starts with unusually large value at $t \to 0$ and decreases with the increase in cosmic time and vanishes as $t \to \infty$.

The Newtonian gravitational constant G and the cosmological constant Λ, for the assumption of $n = 2$, assume the forms

$$G = -\left[\frac{(\mu^2+2\mu)\chi_1^2 e^{\chi t^2}}{2\pi\rho_0\chi} + \frac{h^2k^2\mu\chi_1 e^{(\chi-2\mu\chi_1)t^2}}{4\pi\rho_0\chi k_1^2\rho_2^{2\mu}}\right] \tag{62}$$

$$\Lambda = \left[\frac{4(\mu^2+2\mu)\chi_1^2}{\chi}\right](1+t^2) - \left[\frac{h^2k^2}{\chi k_1^2\rho_2^{2\mu}}\right](\chi - 2\chi_1\mu)e^{-2\mu\chi_1 t^2} \tag{63}$$

where the possible integration constant, for the sake of simplicity, is considered to be zero. Even though it is clear from (62) and (63) that, both G and Λ vary with time, their evolutions have much complicated dependence on time.

The critical densities ρ_c and ρ_Λ for this model can be evaluated as

$$\rho_c = -\frac{2}{3}\left[\frac{(2\mu+1)^2\chi_1 k_1^2\rho_2^{2\mu}\chi\rho_0 t^2}{2(\mu^2+2\mu)\chi_1 k_1^2\rho_2^{2\mu}e^{\chi t^2} + \mu h^2k^2 e^{(\chi-2\mu\chi_1)t^2}}\right] \tag{64}$$

and

$$\rho_\Lambda = -\left[\frac{4(\mu^2+2\mu)\chi_1^2 t^2 - (\frac{h^2k^2}{\chi k_1^2\rho_2^{2\mu}})(\chi - 2\mu\chi_1)e^{(-2\mu\chi_1 t^2)}}{\frac{4(\mu^2+2\mu)\chi_1^2}{\chi\rho_0}e^{\chi t^2} - \frac{2\mu h^2k^2\chi_1}{\chi\rho_0 k_1^2\rho_2^{2\mu}}e^{(\chi-2\mu\chi_1)t^2}}\right] \tag{65}$$

4 Summary and Conclusion

In the present work, we have investigated different bulk viscous anisotropic cosmological models in Einstein's relativity with variable gravitational and cosmological constants. The time varying nature of the cosmological constant is very much linked to the time varying nature of the gravitational constant. If we assume that Λ is a pure constant then as we have shown, G has to be a constant quantity in a matter field universe. In the present work, instead of assuming any particular form of time variation of G and Λ, we have tried to extract their time varying nature from different bulk viscous cosmological models. In getting viable models, we have assumed that the contribution of the bulk viscosity to the total pressure is proportional to the rest energy density. Such an assumption of bulk viscous pressure (may be termed as barotropic bulk viscous pressure) results in a total effective pressure which encompasses real values both in the positive and negative domain. Negative domain pressure is required to explain the observed accelerated expansion of the universe. However, in our models investigated in the present work, the equation of state $\bar{p} = -\rho$ i.e. $\omega = -1$ leads to very unusual results and hence our model is not suitable for that particular value of ω. In order to get some useful information about the time evolution of G and Λ, we have assumed an exponentially decaying energy density of the universe. We have modeled the anisotropic universe through a generalized Bianchi-III metric.

The time varying nature of the Newtonian gravitational constant G and the cosmological constant Λ depends on the bulk viscosity, the anisotropic parameter μ and the form of the decaying energy density. Even though G varies with time, the exact nature of the time variation of G cannot be ascertained as its behavior is more intimately involved with the bulk viscosity, the anisotropic nature of the model and the form of energy density. In other words the time varying nature of these so called constants depends on the factors μ, n, χ and ω. However, for the representative values $n = \frac{1}{2}$, and $n = 1$, G decreases with time.

The cosmological constant is also a decreasing function of time. It assumes a positive and small value as the time grows, which is a bare necessity to explain the present day observations of the accelerated expansion of the universe. Bulk viscosity affects the time varying nature of the cosmological constant. For the models, Λ decreases with time for $n = \frac{1}{2}$, $n = 1$ but has a complicated time dependence involving all the parameters μ, n, χ and ω for $n = 2$.

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